Program Analysis for Quantified Information Flow

The 5th CREST Open Workshop

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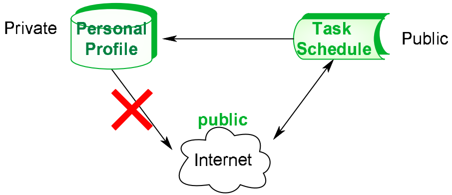
Outline

What is secure information flow?



- Information flows between objects of a computing systems, *e.g.*, devices, agents, variables, channels etc.
- Information flow security is concerned with how security information is allowed to flow through a computer system.
- Flow is considered secure if it accepts a specified policy which defines the accessibility of the information.

Example: Secure information flow is violated

Security level

x: HIGH security variable $\qquad$ y: LOW security variable

Assignment

$y := x;$

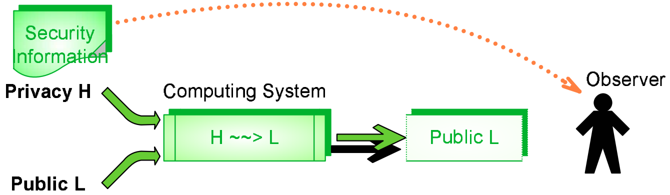
Control flow

$\textit{if } (x \bmod 2 == 0) \textit{ then } y := 0 \textit{ else } y := 1$

Termination behaviour

$y := x;$
$\textit{while}(y \neq 0)\ x := x * x$

Non-interference is too restrictive!



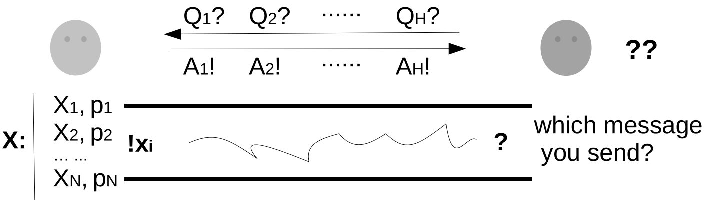
How much information is leaked?

- A new policy to relax the NI
- From quantitative view, the program is secure if the amount of information flow from high to low is small enough.
- Idea: we treat the program as a communication channel, use information theory, consider how much interference?

Outline

Information

An Intuitive Example



- Let H be the average minimum number of questions the receiver needs to guess which symbol you will send:

$$2^H = N \qquad H = \log_2 N$$

$$H = -\log_2 \frac{1}{N} \qquad H = -\log_2 p$$

Information

Information and entropy

- Surprise of an event x_i occurring with probability p_i:

$$-\log_2 p_i$$

- Information (entropy) = expected value of surprise:

$$\mathcal{H} \stackrel{\text{def}}{=} \sum_1^n p_i \log_2 \frac{1}{p_i}$$

- Equivalent to a measurement of uncertainty or variation
- Information is maximised under uniform distribution:

$$\mathcal{H} \leq \log_2 n$$

Random Variables

- A discrete random variable is a surjective function from sample space to observation space:

$$X : D \to \mathcal{R}(D)$$

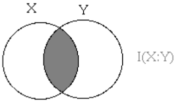
  where D is a finite set with a specified probability distribution, and $\mathcal{R}$ is the finite range of X
- Joint random variable: $\langle X, Y \rangle$
- Random variable X conditioned on $Y = y$: $P(X = x | Y = y)$

Shannon's measure of entropy

Entropy (expected value of surprise when X is observed)

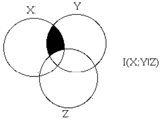
$$\mathcal{H}(X) = \sum_{x \in X} p(x) \log_2 \frac{1}{p(x)} = -\sum_x p(x) \log_2 p(x)$$

Mutual Information (shared information)



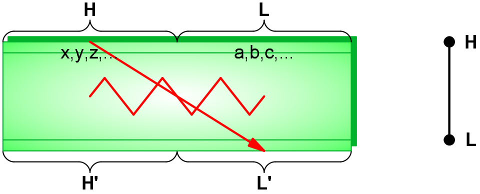
$$\mathcal{I}(X;Y) = \mathcal{H}(X) + \mathcal{H}(Y) - \mathcal{H}(X,Y)$$

Conditional Mutual Information



$$\mathcal{I}(X;Y|Z) = \mathcal{H}(X|Z) + \mathcal{H}(Y|Z) - \mathcal{H}(X,Y|Z)$$

Leakage definition



Leakage Definition for Batch Programs

- $\mathcal{L}(H, L') \triangleq \mathcal{I}(H; L'|L) = \mathcal{H}(L'|L)$ [CHM07]
- Technical considerations allow us to consider $\mathcal{L}(H, L')$ as $\mathcal{H}(L')$ [CHM07]
- How to calculate $\mathcal{H}(L')$??

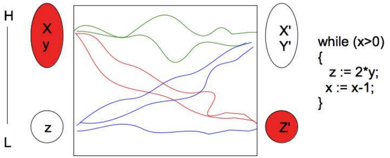
Outline

Current approaches

	description	language	tool	scalability	automatic
Clark,Hunt,Mal	bounds analysis	while	-	$\surd$	$\surd$
Malacaria	partition property	-	-	-	-
McCament,Ernst	dynanmic analysis	C	$\surd$	$\surd$	$\surd$
Backes,Köpf,Ryb	model checking	C	$\surd$	-	$\surd$
Heusser,Mal	model checking	C	$\surd$	-	$\surd$
Lowe	refusal counting	CSP	-	-	-
Boreale	IT in process calculus	CCS	-	-	-

Outline

The idea



- Consider simple imperative programs:
  `skip|ass|if|while|compose`
- Apply probabilistic domain transformer semantics to calculate distribution on outputs given distribution on inputs
- Use information theory to measure flow for a giving input distribution
- Automate the computation of the flows using the semantics

The semantics

$$\begin{array}{ll} \mathcal{M}[\![\texttt{Cmd}]\!] : \Sigma \to \Sigma & \mathcal{M}[\![\texttt{Exp}]\!] : \Sigma \to \mathit{Val} \\ \mathcal{M}[\![\texttt{BExp}]\!] : \Sigma \to \Sigma & \mathit{Val} : X \\ \text{stores } \Sigma : \texttt{Ide} \to \mathit{Val} & \end{array}$$

Figure: Semantics Domains

$$\begin{array}{rcl} f_{[\![x:=e]\!]}(\mu) & \triangleq & \lambda X.\mu(f^{-1}_{[\![x:=e]\!]}(X)) \\ f_{[\![c_1]\!];[\![c_2]\!]}(\mu) & \triangleq & f_{[\![c_2]\!]} \circ f_{[\![c_1]\!]}(\mu) \\ f_{[\![\texttt{if b c}_1\texttt{ c}_2]\!]}(\mu) & \triangleq & f_{[\![c_1]\!]} \circ f_{[\![b]\!]}(\mu) \; + \; f_{[\![c_2]\!]} \circ f_{[\![\neg b]\!]}(\mu) \\ f_{[\![\texttt{while b do c}]\!]}(\mu) & \triangleq & f_{[\![\neg b]\!]}(\lim_{n\to\infty}(\lambda\mu'.\mu + \\ & & f_{[\![c]\!]} \circ f_{[\![b]\!]}(\mu'))^n(\lambda X.\bot)) \\ & & \text{where, } f_{[\![B]\!]}(\mu) = \lambda X.\mu(X \cap B) \end{array}$$

Figure: Probabilistic Denotational Semantics

The leakage definition of loops

Entropy of loops

- We define the leakage for loops up to k^{th} iterations by:

$$\begin{array}{rcl} E & \mapsto & \mathcal{L}_{\texttt{while}}(k) = \widetilde{\mathcal{H}}(\mathcal{P}) + \widetilde{\mathcal{H}}(\mathcal{Q}|\mathcal{P}) \\ & = & \widetilde{\mathcal{H}}(\mathcal{P}_0 \cup \cdots \cup \mathcal{P}_k) + \widetilde{\mathcal{H}}(\mathcal{Q}_0 \cup \cdots \cup \mathcal{Q}_k | \mathcal{P}_0 \cup \cdots \cup \mathcal{P}_k) \end{array}$$

- case $k < n$, we can compute the leakage due to each iteration before the loop terminates with the time of observation
- case $k = n$, this definition has been proved equivalent to Malacaria's leakage definition of loops [Mal07]
- case $k = \infty$, nonterminating loops, $\mathcal{H}(\perp) = 0$

Leakage Analysis by Probabilistic Semantics: Example

Example: A terminating loop

```
l:=0; while(l<h) l:=l+1;
```

- Assume h is 3-bit *high* security variable with distribution:
  $\left[\ 0 \ w.p.\ \frac{7}{8} \quad 1 \ w.p.\ \frac{1}{56} \quad \ldots \quad 7 \ w.p.\ \frac{1}{56}\ \right]$
- l is *low* security variable
- Consider the decompositions $\mathcal{P}_i$ and $\mathcal{Q}_i$ due to event b^i:

$$
\begin{array}{ll}
\mathcal{P}_0 = \{\mu(b^0)\} = \{\frac{7}{8}\} & \qquad \mathcal{Q}_0 = \{\mu_l(0)\} = \{\frac{7}{8}\} \\
\mathcal{P}_1 = \{\mu(b^1)\} = \{\frac{1}{56}\} & \qquad \mathcal{Q}_1 = \{\mu_l(1)\} = \{\frac{1}{56}\} \\
\ldots & \qquad \ldots \\
\mathcal{P}_7 = \{\mu(b^7)\} = \{\frac{1}{56}\} & \qquad \mathcal{Q}_7 = \{\mu_l(7)\} = \{\frac{1}{56}\}
\end{array}
$$

Leakage Analysis by Probabilistic Semantics

Example: A terminating loop

- Note that $q_i = p_i$, hence $\widetilde{\mathcal{H}}(\mathcal{Q}|\mathcal{P}) = 0$, i.e., the information flow within body is 0
- The leakage computation due to each iteration:

$$
\begin{aligned}
\mathcal{L}_{\texttt{while}-0} &= \widetilde{\mathcal{H}}(\mathcal{P}_0) = 0.192645 \\
\mathcal{L}_{\texttt{while}-1} &= \widetilde{\mathcal{H}}(\mathcal{P}_0 \cup \mathcal{P}_1) = 0.304939275 \\
\mathcal{L}_{\texttt{while}-2} &= \widetilde{\mathcal{H}}(\mathcal{P}_0 \cup \mathcal{P}_1 \cup \mathcal{P}_2) = 0.412829778 \\
\mathcal{L}_{\texttt{while}-3} &= \widetilde{\mathcal{H}}(\mathcal{P}_0 \cup \mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3) = 0.516570646 \\
\mathcal{L}_{\texttt{while}-4} &= \widetilde{\mathcal{H}}(\mathcal{P}_0 \cup \mathcal{P}_1 \cup \cdots \cup \mathcal{P}_4) = 0.616396764 \\
\mathcal{L}_{\texttt{while}-5} &= \widetilde{\mathcal{H}}(\mathcal{P}_0 \cup \mathcal{P}_1 \cup \cdots \cup \mathcal{P}_5) = 0.71252562 \\
\mathcal{L}_{\texttt{while}-6} &= \widetilde{\mathcal{H}}(\mathcal{P}_0 \cup \mathcal{P}_1 \cup \cdots \cup \mathcal{P}_6) = 0.805158879 \\
\mathcal{L}_{\texttt{while}-7} &= \widetilde{\mathcal{H}}(\mathcal{P}_0 \cup \mathcal{P}_1 \cup \cdots \cup \mathcal{P}_7) = 0.894483808
\end{aligned}
$$

Outline

The idea

- define an abstraction on the measure space
  - concrte lattice
  - abstract lattice
  - Galois connection
- abstract semantic operations are applied to the abstract space
  - soundness and correctness of the abstraction
- estimate the abstract spaces to provide safe bounds on the entropy computation

Measurable partitions and abstract domain

Concrete lattice

- the σ-algebra $\mathcal{B}$ of a finite measure space X forms a complete lattice
- we define a partial order on $\mathcal{B}$ as follows:

$$\forall x_1, x_2 \in \mathcal{B}, x_1 < x_2 \textit{ iff } \mathcal{H}(x_1) \leq \mathcal{H}(x_2)$$

- define an equivalence relation on $\mathcal{B}$:

$$x_1 \simeq x_2 \textit{ iff } \mathcal{H}(x_1) = \mathcal{H}(x_2)$$

Measurable partitions and abstract domain

Abstract space

- An element of the abstract domain $x_i^\sharp \in X^\sharp$ is defined as a pair $(\mu_i, [E_i])$, where μ_i is the weight on the element
- Adjust the concrete space to be sorted
- Make the partition: $\xi = \{E_i | 1 \leq i \leq n\}$
- Lift to interval-based partition:

$$[E_i] : \langle I_1^i, I_2^i, \ldots, I_k^i \rangle \rightarrow \mu_i$$

Measurable partitions and abstract domain

The Galois connection

- the abstraction function α is a mapping from concrete space X to the sets of interval-based partitions $X^\sharp$: $X \longrightarrow [X/\xi]$, where $[X/\xi] = \{(\mu_i, [E_i]) | 0 < i \leq n\}$
- the concretisation function γ is a mapping: $X^\sharp \rightarrow \bigcup\{x | x \in [E_i]/\eta\}$, where the $[E_i]$ are the blocks of the abstract object $X^\sharp$, η is a sub-partition on each block under uniform distribution

Entropy of Measurable Partition and Leakage Computation

- Uniformalisation: a transformation of each block of space of a variable into a space with uniform distribution on each block
- let $[\![\cdot]\!]\xi = \xi'$, the leakage upper bound

$$
\begin{aligned}
U_v &= \mathcal{H}(\xi'\eta) = \mathcal{H}(\xi') + \mathcal{H}(\eta|\xi') \\
&= \mathcal{H}(\mu_1, \ldots, \mu_n) + \sum_{i=1}^{n} \mu_i \mathcal{H}(\frac{\mu_i/N_i}{\mu_i}, \ldots, \frac{\mu_i/N_i}{\mu_i}) \\
&= \mathcal{H}(\mu_1, \ldots, \mu_n) + \sum_{i=1}^{n} \mu_i \log_2(N_i)
\end{aligned}
$$

where N_i is the size of the partition E_i

Example

$[\![$`l:=0; while(l<h) do l++;`$]\!]$

- initial distribution $\mu_h \mapsto \begin{pmatrix} 0 \ w.p.\ 0.1, & 1 \ w.p.\ 0.1 \\ 2 \ w.p.\ 0.1, & 3 \ w.p.\ 0.1 \\ 4 \ w.p.\ 0.2, & 5 \ w.p.\ 0.2 \\ 6 \ w.p.\ 0.1, & 7 \ w.p.\ 0.1 \end{pmatrix}$
- Consider the partitions ξ:

$$\left\{ \begin{array}{l} E_1\langle [0,3]_h, [0,0]_l \rangle \rightarrow 0.4, \\ E_2\langle [4,7]_h, [0,0]_l \rangle \rightarrow 0.6 \end{array} \right\}$$

Example

$[\![$`l:=0; while(l<h) do l++;`$]\!]$

- A fixpoint is reached at the end.
- Concentrate on the low variable, do uniformalisation on each block to concretise the final space, and have:

$$\left\{\begin{array}{l} [0,3]_l \rightarrow 0.4, \\ [4,7]_l \rightarrow 0.6 \end{array}\right\} \xRightarrow{\texttt{Uniformalisation}} \mu_l \mapsto \left(\begin{array}{cc} 0 \rightarrow 0.4/4 & 1 \rightarrow 0.4/4 \\ 2 \rightarrow 0.4/4 & 3 \rightarrow 0.4/4 \\ ----- & ----- \\ 4 \rightarrow 0.6/4 & 5 \rightarrow 0.6/4 \\ 6 \rightarrow 0.6/4 & 7 \rightarrow 0.6/4 \end{array}\right)$$

- leakage upper bound:
  $U_l = \mathcal{H}(0.4, 0.6) + 0.4 * \log_2 4 + 0.6 * \log_2 4 = 2.97$
- exact leakage: $\mathcal{L} = 2.92$

Outline

The idea

- Consider the quantity of information flow in reactive processes by looking at the different behaviours of a high user from a low user's observations.
- The reactive system is modelled by using Probabilistic Labelled Transition System (PLTS)
- The observation records the history traces of behaviours from the view of low users in a way of distributions.
- Introduce a transformation on the process tree.
- A metric space is built upon the transformation tree, and the information flow is measured via metrics.

The Probabilistic Model

Probabilistic Labelled Transition Systems

- The PLTS is given as a triple $PLTS = (T, \Sigma, \mu)$
- Specifically, $\mu_{p,a} : T \rightarrow [0, 1]$, $\mu_p : \Sigma \rightarrow T \rightarrow [0, 1]$, where for any $a \in \Sigma$ and p is a state that can perform the action a, indicating the possible next states and their probabilities after p has performed a.
- Furthermore, $\forall p \in T$ and can perform action a, $\sum_{p' \in T} \mu_{a,p}(p') = 1$, *i.e.*, $\mu_{p,a}$ is a probability distribution.

The Language and its Semantics

Syntax

$$F ::= \bot \mid x \mid \sum_{i \in I} a_i.p_i.F_i \mid \sqcap S \mid F_1 \| F_2 \mid \mu x.F$$

Operational Semantics

Act $$\frac{E \xrightarrow{a_i}_{p_i} E_i}{E \xrightarrow{a}_{\pi} \sum_{i=1}^{n} p_i.a_i.E_i}$$
$$\pi : \{p_i | 1 \leq i \leq n\},\ a = \{a_i | 1 \leq i \leq n\}$$

Par $$\frac{E_1 \xrightarrow{\tau} E_1'}{E_1 \| E_2 \xrightarrow{\tau} E_1' \| E_2} \qquad \frac{E_2 \xrightarrow{\tau} E_2'}{E_1 \| E_2 \xrightarrow{\tau} E_1 \| E_2'}$$
$$\frac{E_1 \xrightarrow{a}_{\pi_1} E_1' \quad E_2 \xrightarrow{a}_{\pi_2} E_2'}{E_1 \| E_2 \xrightarrow{a}_{\pi_1, \pi_2} \pi_1 \pi_2 . a.(E_1' \| E_2')} (a \neq \tau)$$

Rec $$\frac{}{\mu x.E \xrightarrow{\tau} E[\mu x.E/x]}$$

Observation on traces

- A set of traces can be extracted from the process tree built by the semantics structure.
- Consider the observation as the sum of the low projection on such traces.
- Information on the projection of the high inputs from the trace can be deduced from these observations.
- Under repeated observation on traces, we can deduce probability distributions on the possible traces.

Probabilistic Low Bi-simulation $\sim_L$

- an extension to the concept of bisimulation
- an equivalence relation on the set of processes $\mathcal{R}$ produced by the PLTS, such that, whenever

  $$E_i \sim_L E_j$$

  the following holds:

  $$\forall S \in \mathcal{R}/\sim_L . E_i \xrightarrow{L}_{\mu} S \Leftrightarrow E_j \xrightarrow{L}_{\mu} S$$

  where $\mathcal{R}/\sim_L$ denotes the set of bisimilar classes of $\mathcal{R}$ under $\sim_L$ and $E_i \xrightarrow{L}_{\mu} S$ if and only if $\mu = \sum\{\mu' | E_i' \in S\}$ and $E_i \xrightarrow{L}_{\mu'} E_i'$.

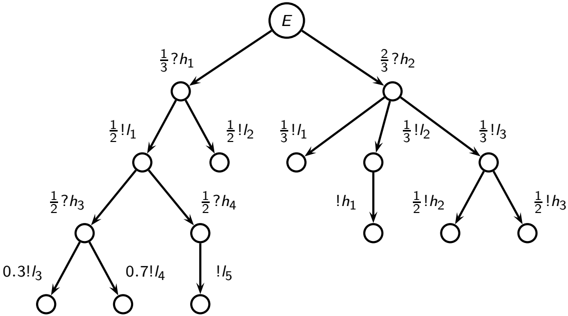
A Transformation on the Process Tree

Interaction unit

Define a (high) interaction unit (step) as a subtree of the process tree whose

- root is labelled by a high input action
- includes every branch terminated by a high input action or $\perp$.

Example process tree



Transformation trees on interaction units

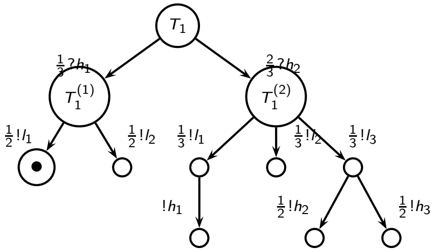


Figure: Transformation tree on the first interaction step T_1

we obtain two subtrees due to the two atomic actions $?h_1$ and $?h_2$ in $?H_0$ as:

$$
\begin{aligned}
T_1^{(1)} &= \left(\frac{1}{2}!l_1.\bot + \frac{1}{2}!l_2.\bot\right) \rightarrow \frac{1}{3} \\
T_1^{(2)} &= \left(\frac{1}{3}!l_1.!h_1.\bot + \frac{1}{3}!l_2.\bot + \frac{1}{6}!l_3.!h_2.\bot + \frac{1}{6}!l_3.!h_3.\bot\right) \rightarrow \frac{2}{3}
\end{aligned}
$$

Transformation trees on interaction units

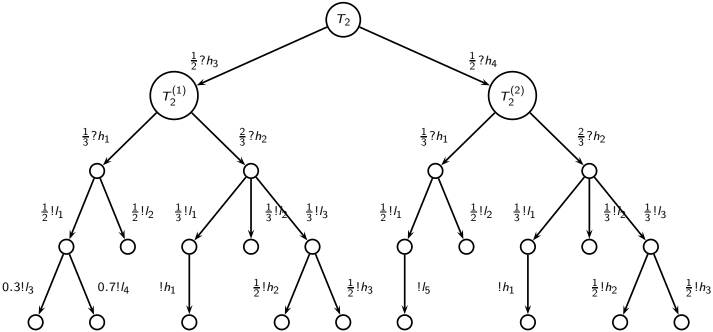


Figure: Transformation tree on the second interaction step T_2

Transformation trees on interaction units

we obtain two subtrees due to the two atomic action $?h_1$ and $?h_2$ in $?H_1$ as:

$$
\begin{aligned}
T_2^{(1)} &= (\frac{0.3}{6}?h_1.!l_1.!l_3.\perp + \frac{0.7}{6}?h_1.!l_1.!l_4.\perp + \frac{1}{6}?h_1.!l_2.\perp + \\
&\quad \frac{2}{9}?h_2.!l_1.!h_1.\perp + \frac{2}{9}?h_2.!l_2.\perp + \frac{1}{9}?h_2.!l_3.!h_2.\perp + \frac{1}{9}?h_2.!l_3.!h_3.\perp) \\
&\quad \rightarrow \frac{1}{2} \\
T_2^{(2)} &= (\frac{1}{6}?h_1.!l_1.!l_5.\perp + \frac{1}{6}?h_1.!l_2.\perp + \frac{2}{9}?h_2.!l_1.!h_1.\perp + \\
&\quad \frac{2}{9}?h_2.!l_2.\perp + \frac{1}{9}?h_2.!l_3.!h_2.\perp + \frac{1}{9}?h_2.!l_3.!h_3.\perp) \rightarrow \frac{1}{2}
\end{aligned}
$$

Observation on the transformation tree

- the observation due to the first interaction unit:

$$\begin{aligned} \mathcal{O}(T_1^{(1)}) &= (\frac{1}{2}!l_1 + \frac{1}{2}!l_2) \to \frac{1}{3} \\ \mathcal{O}(T_1^{(2)}) &= (\frac{1}{3}!l_1 + \frac{1}{3}!l_2 + \frac{1}{3}!l_3) \to \frac{2}{3} \end{aligned}$$

- the observation due to the second interaction unit:

$$\begin{aligned} \mathcal{O}(T_2^{(1)}) &= (\frac{0.3}{6}?h_1.!l_1.!l_3.\perp + \frac{0.7}{6}?h_1.!l_1.!l_4.\perp + \frac{1}{6}?h_1.!l_2.\perp + \frac{2}{9}?h_2.!l_1.\perp + \\ &\quad \frac{2}{9}?h_2.!l_2.\perp + \frac{2}{9}?h_2.!l_3.\perp) \to \frac{1}{2} \\ \mathcal{O}(T_2^{(2)}) &= (\frac{1}{6}?h_1.!l_1.!l_5.\perp + \frac{1}{6}?h_1.!l_2.\perp + \frac{2}{9}?h_2.!l_1.\perp + \frac{2}{9}?h_2.!l_2.\perp + \\ &\quad \frac{2}{9}?h_2.!l_3.\perp) \to \frac{1}{2} \end{aligned}$$

Information Flow Measurement

Jensen-Shannon Divergence (JSD)

- Consider m distributions $P^{(1)}, P^{(2)}, \ldots, P^{(m)}$.
- The JSD between the m distributions $P^{(1)}, \ldots, P^{(m)}$ with weights $w^{(1)}, \ldots, w^{(m)}$ is given by

$$D_{\mathrm{JS}}(P^{(1)}, P^{(2)}, \ldots, P^{(m)}) = \mathcal{H}(\sum_{j=1}^{m} w^{(j)} P^{(j)}) - \sum_{j=1}^{m} w^{(j)} \mathcal{H}(P^{(j)})$$

Information Flow Measurement

The Metric

For a set of processes $f_1, \ldots, f_m \in \mathcal{R},\ d_\mu(f_1, \ldots, f_m)$ is defined as:

$$d_\mu(f_1, \ldots, f_m) = \sqrt{\mathcal{H}(\sum_{j=1}^{m} w^{(j)} P^{(j)}) - \sum_{j=1}^{m} w^{(j)} \mathcal{H}(P^{(j)})}$$

Proposition 2.

For any processes $f_1, \ldots, f_m \in \mathcal{R},\ d(f_1, \ldots, f_m),\ d(f_1, \ldots, f_m) = 0$ iff $f_1 \sim_L \cdots \sim_L f_m$.

Build the metric spaces

- Consider all the interaction units, we build a collection of metric spaces $(\bigcup T_i, \bigcup d_i),\ (i = 0, 1, \dots)$:

$$T_0 = \{p_0\},\ T_1 =?H_0 \prec T_0,\ \dots,\ T_{n+1} =?H_n \prec T_n$$

- Clearly, for $T_i^{(1)}, \dots, T_i^{(m_i)} \in T_i$,
  - if $P_i^{(1)} = \cdots = P_i^{(m_i)},\ d_i = 0$;
  - otherwise $d_i = \sqrt{D_{\mathrm{JS}}(P_i^{(1)}, \dots, P_i^{(m_i)})}$ is the metric between the distributions extracted from the subtree set due to the interaction step i.

Quantity of the information flow

Definition of leakage

- For each interaction unit started by

$$?H_{i-1} = \{?h_{i-1,1} \to w_{i-1,1}, \ldots, ?h_{i-1,m_{i-1}} \to w_{i-1,m_{i-1}}\}$$

we have built a metric space $(T_i, d_i)_{i \geq 1}$, where

$$d_i = \sqrt{D_{\mathrm{JS}}(P_i^{(1)}, \ldots, P_i^{(m_{i-1})})}$$

- The leakage upper bound is defined as the square of the sum:

$$(\sum_{i=1}^{n} d_i)^2$$

Quantity of the information flow

Example

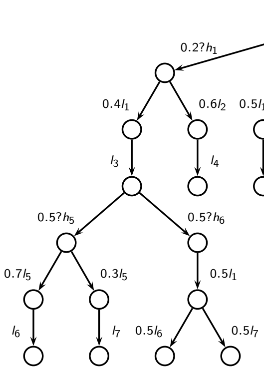


Figure: The example process tree

Quantity of the information flow

Example

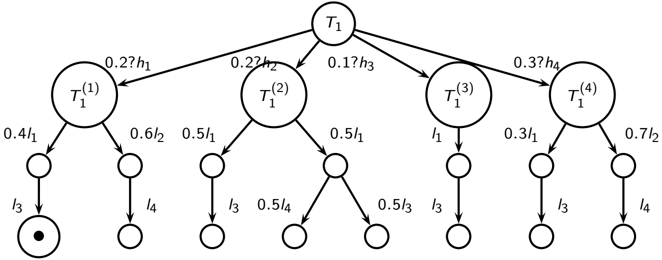


Figure: Transformation on the first interaction unit: T_1

Quantity of the information flow

Example

- the observations:

$$\begin{aligned}
\mathcal{O}(P_0^{(1)}) &= 0.4 \cdot l_1.l_3.\perp + 0.6 \cdot l_2.l_4.\perp \\
\mathcal{O}(P_0^{(2)}) &= 0.5 \cdot l_1.l_3.\perp + 0.25 \cdot l_2.l_4.\perp + 0.25 \cdot l_2.l_3.\perp \\
\mathcal{O}(P_0^{(3)}) &= l_1.l_3.\perp \\
\mathcal{O}(P_0^{(4)}) &= 0.3 \cdot l_1.l_3.\perp + 0.7 \cdot l_2.l_4.\perp
\end{aligned}$$

- the metric:

$$\begin{aligned}
d_1 &= \sqrt{\mathcal{H}(\sum_{i=1}^{4} w_0^{(i)} P_0^{(i)}) - \sum_{i=1}^{4} w_0^{(i)} \mathcal{H}(P_0^{(i)})} \\
&= \mathcal{H}(0.47, 0.43, 0.1) - (0.2\mathcal{H}(0.4, 0.6) + 0.4\mathcal{H}(0.5, 0.25, 0.25) + \\
&\quad 0.1 * 0 + 0.3\mathcal{H}(0.3, 0.7)) = 0.557
\end{aligned}$$

Quantity of the information flow

Example

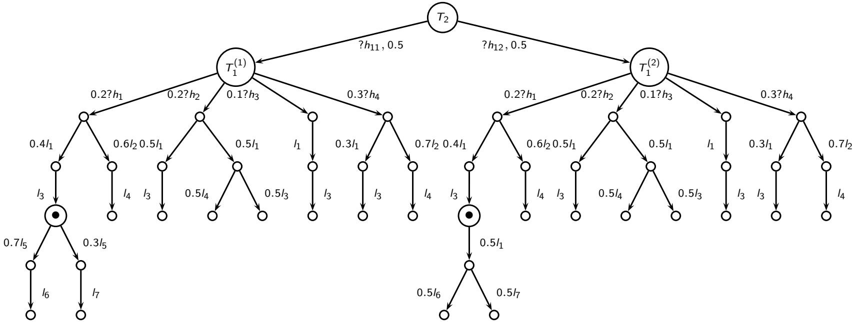


Figure: Transformation on the second interaction: T_2

Example: quantity of the information flow

- observations:

$$
\begin{aligned}
\mathcal{O}(P_1^{(1)}) &= 0.004?h_1.l_1.l_3.l_5.l_6.\perp + 0.04?h_1.l_1.l_3.l_5.l_7.\perp \\
&\quad 0.12?h_1.l_2.l_4.\perp + 0.2?h_2.l_1.l_3.\perp + 0.1?h_2.l_2.l_4.\perp + 0.1?h_2.l_2.l_3.\perp \\
&\quad 0.1?h_3.l_1.l_3.\perp + 0.09?h_4.l_1.l_3 + 0.21?h_4.l_2.l_4
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{O}(P_1^{(2)}) &= 0.056?h_1.l_1.l_3.l_5.l_6.\perp + 0.024?h_1.l_1.l_3.l_5.l_7.\perp \\
&\quad 0.12?h_1.l_2.l_4.\perp + 0.2?h_2.l_1.l_3.\perp + 0.1?h_2.l_2.l_4.\perp + 0.1?h_2.l_2.l_3.\perp \\
&\quad 0.1?h_3.l_1.l_3.\perp + 0.09?h_4.l_1.l_3 + 0.21?h_4.l_2.l_4
\end{aligned}
$$

- the metric:

$$
\begin{aligned}
d_2 &= \sqrt{\mathcal{H}(\sum_{i=1}^{2} w_1^{(i)} P_1^{(i)}) - \sum_{i=1}^{2} w_1^{(i)} \mathcal{H}(P_1^{(i)})} \\
&= (\mathcal{H}(0.048, 0.032, 0.12, 0.2, 0.1, 0.1, 0.1, 0.09, 0.21) - \\
&\quad 0.5\mathcal{H}(0.056, 0.024, 0.12, 0.2, 0.1, 0.1, 0.1, 0.09, 0.21) - \\
&\quad 0.5\mathcal{H}(0.04, 0.04, 0.12, 0.2, 0.1, 0.1, 0.1, 0.09, 0.21))^{\frac{1}{2}} \doteq 0.049
\end{aligned}
$$

- The leakage upper bound:

$$\mathcal{L} \leq (d_1 + d_2)^2 \doteq 0.36$$

Outline

Conclusions

- We present an automatic analysis for measuring information flow within software systems.
- We quantify leakage in terms of information theory and incorporate this computation into probabilistic semantics.
- An abstraction on the exact leakage analysis
- An approach for leakage analysis in reactive processes