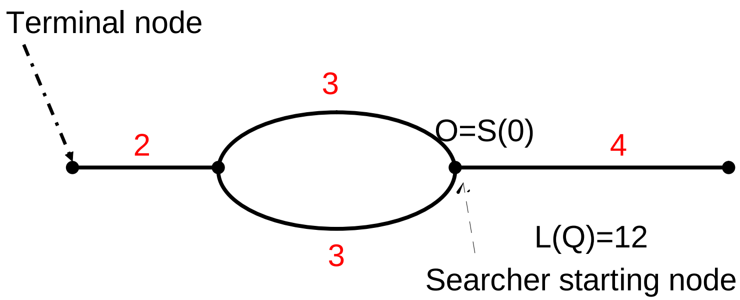
Minimax Search of a Network

Steve Alpern

Department of Mathematics, LSE

Search for Immobile Hider on a Network

Every edge e of Q has a length $L(e)$ and the total length is denoted by $L(Q) = \mu$. The length of a minimal (Chinese Postman) tour is denoted $\bar{\mu}$.



Bounds on V=V(Q,O) for a General Network

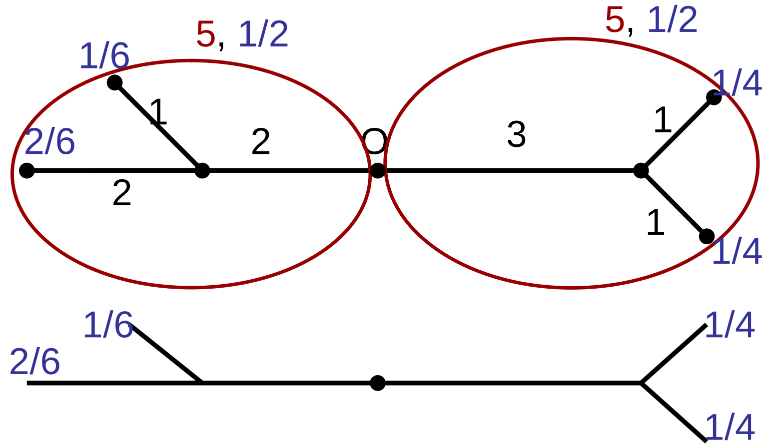
Theorem (Gal): For any network (Q, O), the value V of the search game for an immobile hider satisfies

$$\frac{\mu}{2} \leq V \leq \frac{\bar{\mu}}{2}.$$

The lower bound holds iff Q is Eulerian (has Eulerian Tour). The upper bound holds for trees and iff Q is Weakly Eulerian (Gal), that is, consists of a disjoint family of Eulerian networks connected in a tree like fashion.

Equal Branch Density (EBD) Hider Distribution on Trees

The optimal Hider distribution on a tree is the EBD distribution e. At every branch node it assigns probabilities to the branches proportional to their lengths.



Arc-Adding Lemma: Get Q' from a Q by adding edge e of length $l \geq 0$ between points $x, y \in Q$. Then

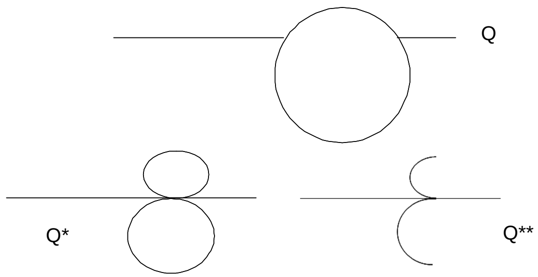
1. $V(Q') \leq V(Q) + 2l$, so $V(Q') \leq V(Q)$ if we identify v_1, v_2 $(l = 0)$

2. If $l \geq d_Q(v_1, v_2)$, then $V(Q') \geq V(Q)$. Any hiding strategy on Q does as well on Q'.

Weakly Eulerian Networks

Definition: A network is weakly Eulerian if it contains a set of disjoint Eulerian networks such that shrinking each to a point transforms the network into a tree.

Proposition (Gal): If Q is a weakly Eulerian network then $V = \bar{\mu}/2$.



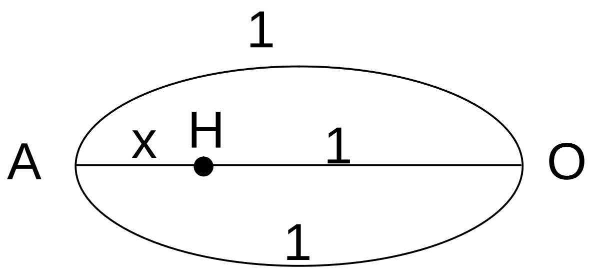
All three networks have same $\bar{\mu}$. $V\left(Q^*\right) \leq V\left(Q\right)$ and $V\left(Q^*\right) \geq V\left(Q^{**}\right)$ Arc-Adding Lemma. $V\left(Q^{**}\right) = \bar{\mu}\ /2)$ (tree).

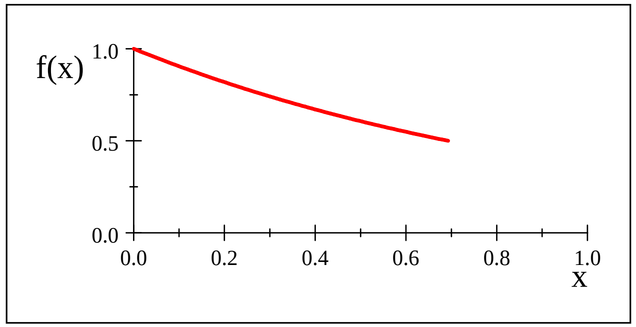
$$\bar{\mu}/2 \leq V\left(Q\right) \leq \bar{\mu}/2.$$

Gal's Theorem

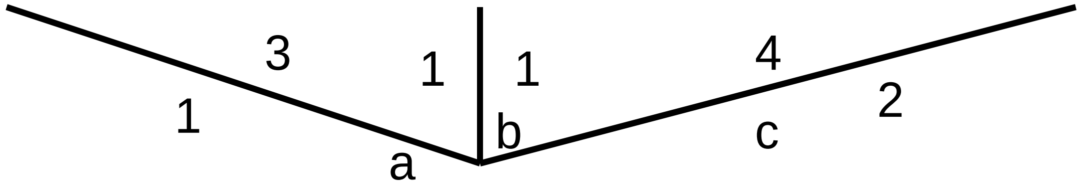
Theorem [3.26]: For any network Q, $V = \bar{\mu}/2$ iff Q is weakly Eulerian.

The 'Three Arc' Network





Best to hide near A. Pick x (on random arc) with probability density $f(x) = e^{-x}$ $0 < x < \ln 2 \approx .693$. Searcher goes to A, back a bit on another arc, back to A, back to O, back towards A. (S. Gal, L. Pavlovic). $V = (4 + \ln 2)/3 \approx 1.56 < \bar{\mu}/2$.



Tree with asymmetric distances (travel times):
out (left) back (right)

$$\text{(Alpern-Lidbetter Formula): } V = \frac{\bar{\mu}}{2} + \frac{1}{2} \sum_{\text{leaves } j} e(j) \quad \left(d(0,j) - d(j,0)\right). \quad (1)$$