# An interactive approach for the Multi-Criteria Portfolio Selection Problem

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LSE and IST

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# Overview

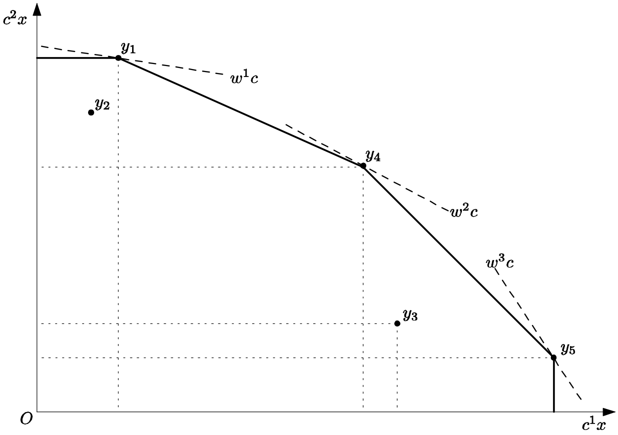
- The standard formulation of the MCPSP.
- New 'integrated' formulations.
- Enumerating efficient Portfolios.
- Incorporating preferences.
- Interactive procedures.

# MOBO formulation of the MCPSP

$$
\begin{aligned}
\text{'max'} \;\; & (c^1 x, \ldots, c^p x) \\
\text{s.t.} \;\; & a^i x \leq b_i \;\; \forall i \in I \\
& x_j \in \{0, 1\} \;\; \forall j \in J.
\end{aligned}
$$

- $J = \{1, \ldots, n\}$ : The set of $n$ projects.
- $c^r$, $r \in R = \{1, \ldots, p\}$ : Objective vectors, assumed non-negative.
- $a^i$, $i \in I = \{1, \ldots, m\}$ : Resource utilisation vectors.
- $b_i$, $i \in I$ : Resource levels.

# The problem



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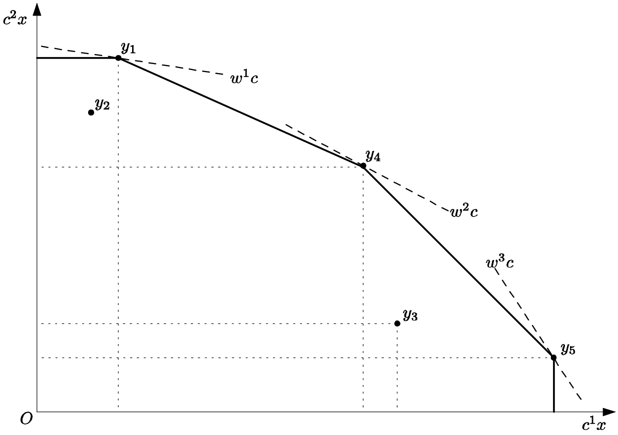


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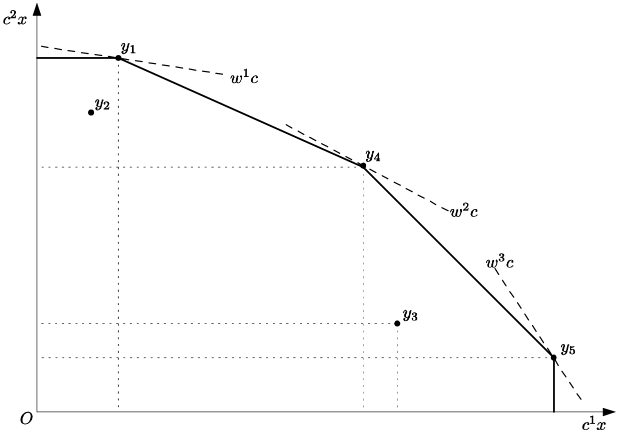


- **Motivation**: Can we identify supported efficient portfolios without selecting weights *a-priori?*
- Integrate criterion weights with binary decision variables in a single optimisation problem.

# Integrated formulations

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\text{s.t.} \; & a^i x \leq b_i \; \forall i \in I \\
& \textstyle\sum_r w^r = 1 \\
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  2. $\sum_r w^r \sum_{j \in \Pi^k} c_j^r \leq \sum_r \sum_j z_j^r c_j^r$
- In this fashion we can enumerate the set of supported efficient portfolios.

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- Preferences can be incorporated by appending $W$ to our formulation.

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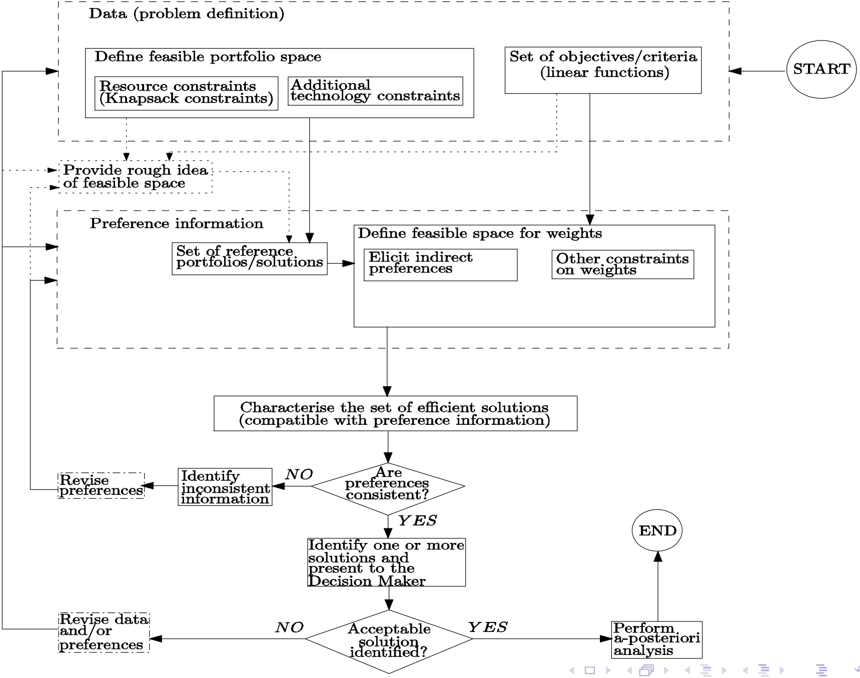
- Solving iteratively, the incumbent solution converges to a *preferred* solution (when $\beta^* = 0$).

# A general interactive scheme


Data (problem definition)
Define feasible portfolio space
Resource constraints (Knapsack constraints)
Additional technology constraints
Set of objectives/criteria (linear functions)
START
Provide rough idea of feasible space
Preference information
Set of reference portfolios/solutions
Define feasible space for weights
Elicit indirect preferences
Other constraints on weights
Characterise the set of efficient solutions (compatible with preference information)
Are preferences consistent?
NO
Identify inconsistent information
Revise preferences
YES
Identify one or more solutions and present to the Decision Maker
Acceptable solution identified?
NO
Revise data and/or preferences
YES
Perform a-posteriori analysis
END

Questions/Comments