Interference, Dependence and Bell's Theorem

Samson Abramsky

Department of Computer Science, University of Oxford

Our Themes

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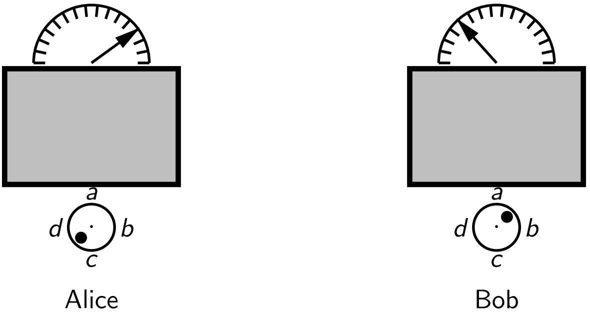
- Interference and dependence are not only pervasive in Computer Science, but throughout the sciences.
- In fact, they play a key rôle in fundamental results such as Bell's theorem in the foundations of quantum mechanics — seminal for subsequent developments in quantum information and computation.
- There is a fascinating interplay between logical structure, probability, observational ideas — with implications for the nature of physical reality!

The Basic Scenario



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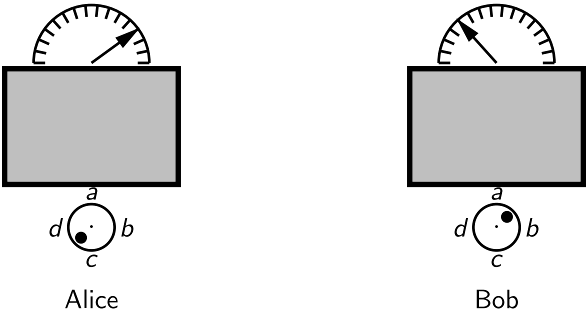

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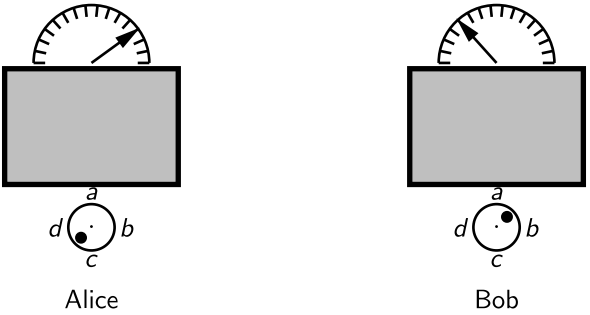

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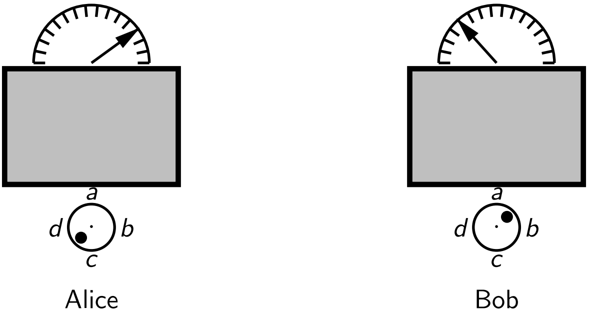

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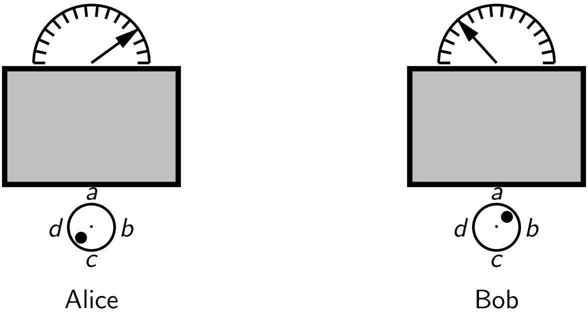

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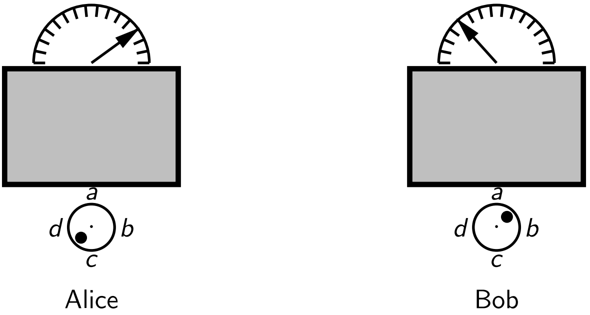

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Cf. 'measurement-based verification'.

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What does reasonable mean??

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$$1 - P = \mathrm{Prob}(\neg\Phi) = \mathrm{Prob}(\bigvee_i \neg\phi_i) \leq \sum_i \mathrm{Prob}(\neg\phi_i) = \sum_i (1 - p_i) = N - \sum_i p_i.$$

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Now suppose that the formulas ϕ_i are **jointly contradictory**; *i.e.* Φ is unsatisfiable. Clearly, we must then have $P = 0$. Hence we obtain the inequality

$$\sum_i p_i \;\leq\; N - 1.$$

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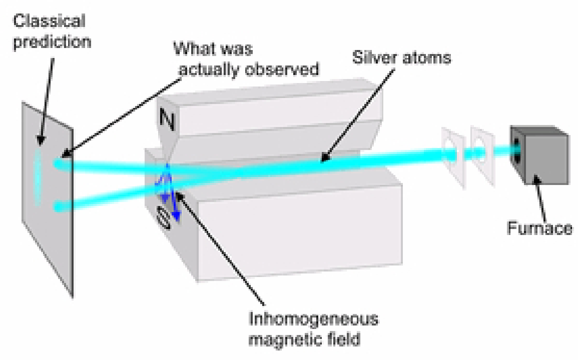
We shall call an inequality

$$\sum_{i=1}^{N} p(\varphi_i) \;\leq\; N-1$$

where $\bigwedge_i \varphi_i$ is unsatisfiable a **logical Bell inequality**.

The Stern-Gerlach Experiment



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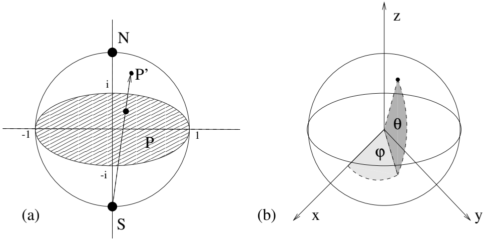

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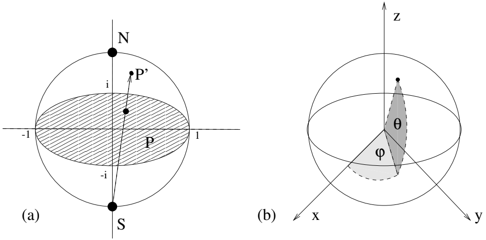

Things get interesting with n-qubit registers

$$\sum_i \alpha_i |i\rangle, \qquad i \in \{0,1\}^n.$$

Quantum Entanglement

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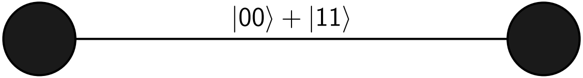

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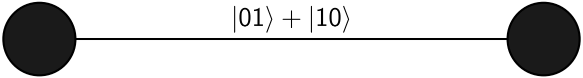

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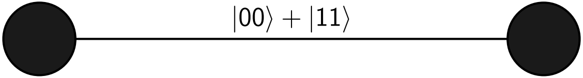

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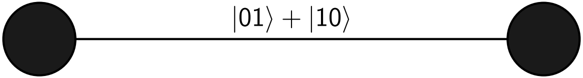

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$$\sum_i \lambda_i \cdot \phi_i \otimes \psi_i$$

Superposition encodes **correlation**.

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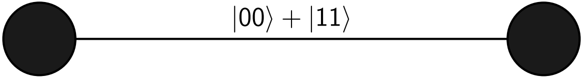

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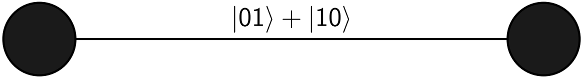

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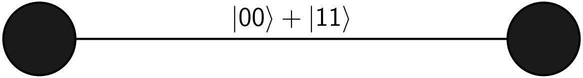

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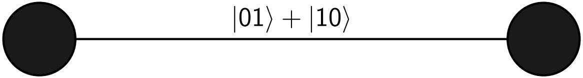

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Bell's theorem: QM is **essentially non-local**.

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Generated by a Bell state

$$\frac{|00\rangle + |11\rangle}{\sqrt{2}},$$

subjected to measurements in the XY-plane, at relative angle $\pi/3$.

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If we read 0 as true and 1 as false, the highlighted positions in each row of the table are represented by the following propositions:

$$\begin{array}{lclclcl}
\varphi_1 & = & a \wedge b & \vee & \neg a \wedge \neg b & = & a \leftrightarrow b \\
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\varphi_4 & = & \neg a' \wedge b' & \vee & a' \wedge \neg b' & = & a' \oplus b'.
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The violation of the logical Bell inequality is 1/4.

Example: the Hardy model

The support of the Hardy model:

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In this model, $p_2 = p_3 = p_4 = 1$.

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However, these formulas are not simultaneously satisfiable.

In this model, $p_2 = p_3 = p_4 = 1$.

Hence the Hardy model achieves a violation of $p_1 = \text{Prob}(a \wedge b)$ for the logical Bell inequality.

Example: the Hardy model

The support of the Hardy model:

	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
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This is 'physical reality' we are talking about!

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Also, since observations act on (interfere with) the system, **we cannot perform all observations together, even in principle**.

This leads to contextual behaviour: it matters which measurement context we are referring to.

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A subtle set sandwiched between two simpler sets (polytopes).

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We shall prove that such models are realizable in QM, using the GHZ states.

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The relevant part of the support table:

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abc	1	0	0	1	0	1	1	0
$ab'c'$	0	1	1	0	1	0	0	1
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Given boolean variables x, y, z, we define

$$\Psi_{xyz} \;:=\; x \oplus y \oplus z.$$

The support for each row can be specified by the following formulas:

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These formulas are not simultaneously satisfiable; thus this model achieves a **maximal violation** of a logical Bell inequality.

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Papers:

- S. Abramsky and A. Brandenburger. The sheaf-theoretic structure of non-locality and contextuality. *New Journal of Physics*, 13(2011):113036, 2011.
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